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Stability properties of a general class of nonlinear dynamical systems

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Abstract

We establish sufficient conditions for the boundedness of the trajectories and the stability of the fixed points in a class of general nonlinear systems, the so-called quasi-polynomial vector fields, with the help of a natural embedding of such systems in a family of generalized Lotka–Volterra (LV) equations. A purely algebraic procedure is developed to determine such conditions. We apply our method to obtain new results for LV systems, by a reparametrization in time variable, and to study general nonlinear vector fields, originally far from the LV format.

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1. Introduction

The study of the global stability of fixed points of a continuous dynamical system is a very important issue in the qualitative analysis as this property implies several restrictions in the evolution of the trajectories (Mclachan *et al* 1998, Figueiredo *et al* 2000). Here, stability is meant in the sense of Lyapunov (1949), where the stability of the fixed point is granted by the existence of a positive-definite function, known as *Lyapunov function*, in a neighbourhood of the fixed point, such that its total time derivative is negative-definite (or semidefinite). Furthermore, the boundedness of the solutions can be established by LaSalle's invariance principle (LaSalle and Lefshets 1961). The major drawback of this approach is the lack of a general prescription for the determination of the Lyapunov function (Haykin 1999), with few exceptions (for example, a monotonously decreasing energy for non-conservative systems). Nevertheless, for quadratic systems of Lotka–Volterra (LV) type, commonly used in the study of population dynamics, a natural candidate for the Lyapunov function is known: see, for example, Takeuchi (1995) and references therein. One of the main purposes of this paper is to show how to extend the results obtained for LV systems to more general dynamical systems,

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the *quasi-polynomial systems*, by recasting the latter in a LV canonical format, as explained below.

In the last 20 years many authors have shown how to cast a wide class of dynamical systems into universal formats, usually by increasing the dimension of the original system (Brenig 1988, Brenig and Goriely 1989). One such format that has drawn some attention is the system of quasi-polynomial (QP) ordinary differential equations (Figueiredo *et al* 1998, 1999). All these formats have in common that they can be related to a quadratic LV system of the type found in models of biological population dynamics systems. The QP format is particularly useful, as its reduction to the LV system is obtained by an embedding plus a quasi-monomial coordinate transformation (see section 3), which allows in a direct way the study of the topology and of the dynamics in the original phase space from those of the associated LV system (Brenig 1988).

The main purposes of this paper are to present a new approach to determine existence conditions for Lyapunov functions in systems of ordinary differential equations of the QP type and to establish the boundedness of its solutions and the invariance of the orthants. The basis of the method consists in the generalization of a theorem, originally concerning LV systems, to encompass the QP systems. This theorem furnishes sufficient conditions for the stability of the fixed points and then our approach delivers a systematic but not exhaustive method for the determination of Lyapunov functions in QP systems. Furthermore, we show how a special reparametrization in the time variable allows us to enlarge the scope of applications. This will lead to new existence conditions of Lyapunov functions in purely LV systems, involving its linear terms, while in the original theorem only the quadratic terms are involved.

This paper is structured as follows. In section 2 we review the results of Redheffer and Walter (1984). In section 3 we present the relationship between the QP systems and the LV systems, and show how to obtain, in a constructive way, a Lyapunov function for a QP system. In section 4 we present an algorithm to establish purely algebraic conditions for the existence of the Lyapunov function. In section 5 we establish new results for the LV systems by a suitable reparametrization in the time variable and we obtain new conditions for stability in the May–Leonard system. In section 6 we present an application for a typical QP system. In section 7 we discuss a numerical extension of our method. Finally, in section 8 we close with some concluding remarks.

2. Stability problem in generalized Lotka–Volterra systems

Here we present some results obtained in Redheffer and Walter (1984). Consider a general LV system given by

$$\dot{U}_i = \lambda_i U_i + U_i \sum_{j=1}^{m} M_{ij} U_j \qquad i = 1, \ldots, m. \tag{1}$$

The variables U_i are real-valued functions and M is a square $m \times m$ matrix. The fixed points of equation (1) in the positive orthant are the solutions of the equations

$$\lambda_i + \sum_{j=1}^{m} M_{ij} q_j = 0 \qquad i = 1, \ldots, m \quad q_i > 0. \tag{2}$$

The study of the asymptotic behaviour of (1) was first considered by Volterra (1931), and other authors have faced this problem since his pioneering work: see Takeuchi (1995) and references therein. In Redheffer and Walter (1984) the concept of *admissible matrix* was introduced to deal with the problem of stability and asymptotic behaviour of LV systems. A

$m \times m$ matrix M is said to be *admissible* if there are constants $a_i > 0$, $i = 1, \ldots, m$, such that

$$\sum_{i,j=1}^{m} a_i M_{ij} w_i w_j \leqslant 0 \qquad w \in \mathbb{R}. \tag{3}$$

This condition implies that $M_{ii} \leqslant 0$, and can hold even in the case of a singular matrix.

The following theorem was proved in Redheffer and Walter (1984).

Theorem 1. *If M given in (1) is admissible and there is a fixed point q_i in the interior of the positive orthant then there exists a Lyapunov function V_q with respect to this fixed point, given by*

$$V_q = \sum_{i=1}^{m} a_i \left(U_i - q_i \ln \frac{U_i}{q_i} - q_i \right) \tag{4}$$

and, furthermore, there exist $2m$ positive numbers ϵ_i, ν_i such that

$$\epsilon_i < U_i(t) < \nu_i \qquad \forall\, t. \tag{5}$$

The existence of a Lyapunov function ensures the stability of the interior fixed points of the LV system, and the LaSalle invariance principle (LaSalle and Lefshets 1961) guarantees the boundedness of the solutions, i.e. equation (5).

Theorem 1 holds even in the case of a singular matrix M, corresponding to the existence of degenerated fixed points. The hypotheses $U_i(0) > 0$, $q_i > 0$, are appropriate in calculations, but as far the mathematical development is concerned, they can be replaced by

$$U_i(0) q_i > 0 \tag{6}$$

and all the previous results remains unchanged.

3. General representation of quasi-polynomial systems as Lotka–Volterra equations

A QP dynamical system can be written as

$$\dot{x}_i = l_i x_i + x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}} \qquad i = 1, \ldots, n \tag{7}$$

with $x_i \in \mathbb{R}^n$, where A and B are real, constant rectangular matrices. The number m is related to the number of monomials in the vector field of equation (7). In Brenig (1988) it was shown that a QP system of form (7) with $m < n$ can always be decomposed into an equivalent system with $n' \leqslant m$ equations. Therefore, in all that follows we consider without loss of generality $m \geqslant n$ in equation (7). We also assume that the rank of B is maximal, i.e. it is equal to n. This represents no restriction at all for the present approach since if the rank is less than n then the system can be decoupled into a new QP system with a new exponent B' whose rank is maximal, see Brenig (1988). The denomination *quasi-monomial* for $\prod (x_k)^{B_{jk}}$ is used as the exponents B_{jk} are allowed to be real numbers. This is also the reason for the denomination of QP differential equations. These equations are ubiquitous in physics, chemistry and biomathematics.

We introduce $m - n$ extra variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ such that

$$\dot{x}_k = 0 \qquad k = n+1, \ldots, n+m \tag{8}$$

and obtain the extended system

$$\dot{x}_i = \tilde{l}_i x_i + x_i \sum_{j=1}^{m} \tilde{A}_{ij} \prod_{k=1}^{m} x_k^{\tilde{B}_{jk}} \qquad i = 1, \ldots, m \tag{9}$$

where

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{10}$$

and

$$\tilde{B} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} & b_{1,n+1} & \cdots & b_{1,m} \\ B_{21} & B_{22} & \cdots & B_{2n} & b_{2,n+1} & \cdots & b_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mn} & b_{m,n+1} & \cdots & b_{m,m} \end{bmatrix}. \tag{11}$$

The parameters b_{jk} are arbitrary with the only restriction that $\tilde{B}$ is invertible. To ensure the equivalence between (7) and (9), the following initial conditions are imposed:

$$x_k(t = 0) = 1 \qquad k = n + 1, \ldots, m. \tag{12}$$

System (9) is covariant under the quasi-monomial transformations

$$x_i = \prod_{\beta=1}^{m} U_\beta^{D_{i\beta}} \tag{13}$$

with D invertible. The inverse of the above transformation is given by

$$U_\alpha = \prod_{i=1}^{m} x_i^{D^{-1}_{\alpha i}}. \tag{14}$$

In particular, equation (9) is embedded into a LV-type equation when D is taken to be $\tilde{B}^{-1}$:

$$\dot{U}_\alpha = (\tilde{B}\tilde{l})_\alpha U_\alpha + U_\alpha \sum_{\beta=1}^{m} (\tilde{B}\tilde{A})_{\alpha\beta} U_\beta \qquad \alpha = 1, \ldots, m. \tag{15}$$

Now let us define the characteristic matrix $M \equiv BA = \tilde{B}\tilde{A}$ and $\lambda \equiv Bl = \tilde{B}\tilde{l}$, which identifies (15) as a LV system of type (1). Therefore, every QP system can be cast into a LV-type system by a suitable embedding defined by equations (9), (12) and the coordinate transformation in equation (13). Using (12) and (13) we have

$$\prod_{\beta=1}^{m} U_\beta^{\tilde{B}^{-1}_{\alpha\beta}} = 1 \qquad \alpha = n + 1, \ldots, m \tag{16}$$

which defines a subspace of the m-dimensional LV space where lies the original dynamics of equation (7). So the LV equations (15), with $M = BA$ and $\lambda = Bl$, restricted to the hypersurfaces defined by (16) are equivalent to the QP system (7). Moreover, there is a one-to-one correspondence between the fixed points of (7) and the points in the intersection of the hypersurfaces (16) with the fixed points of (15), provided that the coordinates of the fixed points are all non-zero. Let us restrict ourselves to fixed points in the interior of the positive orthant. We state the following theorem.

Theorem 2. *If the matrix $M = BA$ associated to the QP system (7) is admissible and there is a fixed point x^* of (7) in the positive orthant, then the Lyapunov function $V_q(U)$ in equation (4) can be used to obtain a Lyapunov function $\overline{V}(x)$ for system (7) from the restriction in equation (12):*

$$\overline{V}(x) = V_q(U)\big|_{x_{n+1}=\cdots=x_m=1} \tag{17}$$

U and x related by (13), with $D = \tilde{B}^{-1}$, and the fixed points q of the associated LV system are related to x^* by

$$q_\alpha = \prod_{i=1}^{n} (x_i^*)^{B_{i\alpha}}. \tag{18}$$

Proof. For $\overline{V}(x)$ to be a Lyapunov function the following conditions must hold (Lyapunov 1949):

(a) $\overline{V}(x^*) = 0$;
(b) $\overline{V}(x) > 0$ if $x \neq x^*$;
(c) $\frac{\mathrm{d}\overline{V}(x)}{\mathrm{d}t} \leqslant 0$.

Properties (a) and (b) are obvious since $\overline{V}(x)$ is a restriction of $V_q(U)$ to a subspace of the m-dimensional manifold of the coordinates U_i, and $V_q(U)$ satisfies (a) and (b). Property (c) is immediate: the Lyapunov function $V_q(U)$ satisfies

$$\frac{\mathrm{d}V_q(U)}{\mathrm{d}t} \leqslant 0 \tag{19}$$

and therefore

$$\frac{\mathrm{d}V_q(U)}{\mathrm{d}t} = \sum_{\alpha=1}^{m} \frac{\partial V_q}{\partial U_\alpha}\frac{\mathrm{d}U_\alpha}{\mathrm{d}t} = \sum_{\alpha=1}^{m}\sum_{i=1}^{n} \frac{\partial V_q}{\partial U_\alpha}\frac{\partial U_\alpha}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\partial V_q(U(x))}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} \leqslant 0. \tag{20}$$

The restriction of (20) by equation (12) gives

$$\sum_{i=1}^{n} \frac{\partial V_q(U(x))}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t}\bigg|_{x_{n+1}=\cdots=x_m=1} = \sum_{i=1}^{n} \frac{\partial \overline{V}(x)}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\mathrm{d}\overline{V}(x)}{\mathrm{d}t} \leqslant 0.$$

The Lyapunov function $\overline{V}$ is defined for all points in the positive orthant. □

Let us now state another important result of this paper.

Theorem 3. *If the matrix $M = BA$ is admissible and the initial condition is in the positive orthant, then the corresponding solution is bounded and componentwise bounded away from zero; that is, if $(x_1(0), \ldots, x_n(0)) \in \mathbb{R}_+^n$, then*

$$\exists\, \epsilon_i, \delta_i \in \mathbb{R} \mid 0 < \epsilon_i < x_i(t) < \delta_i \qquad i = 1, \ldots, n \quad \forall\, t.$$

Proof. Let us consider the quasi-monomials in equation (7):

$$U_i = \prod_{k=1}^{n} x_k^{B_{ik}} \qquad i = 1, \ldots, m. \tag{21}$$

We reorder the quasi-monomials in (21) such that the first n lines of matrix B are linearly independent. This is possible as B is of rank n. Let us define a new $n \times n$ matrix $\hat{B}$ by

$$\hat{B}_{ij} = B_{ij} \qquad i, j = 1, \ldots, n. \tag{22}$$

The n first quasi-monomials can then be written as

$$U_i = \prod_{k=1}^{n} x_k^{\hat{B}_{ik}}. \tag{23}$$

This transformation of variables can be inverted as $\hat{B}$ is non-singular:

$$x_i = \prod_{k=1}^{n} U_k^{\hat{B}^{-1}_{ik}}. \tag{24}$$

From theorem 1 each term in the right-hand side of equation (24) is bounded from above and from below and therefore the same holds for x_i. This closes the proof. □

Theorem 4 states sufficient conditions on A and B for the invariance of the positive orthant, which is far from being a trivial result for QP systems. In fact, the theorem ensures that the solutions are away from the coordinate planes $x_i = 0$ and bounded.

As for the LV equations, the restriction to the positive orthant is not necessary. In fact, if the QP system in equation (7) is defined in other orthants, the above results are still valid provided the initial condition and the fixed point are in the same orthant.

The conditions on the statements of theorems 2 and 3 can be somewhat relaxed as follows. Let us suppose once again that the quasi-monomial functions U_i in equation (21) are ordered in such a way that the first lines of B are linearly independent. This ensures the inversibility of the transformation in equation (23), as explained above. Then the condition $a_i > 0$ in equation (3) can be replaced by $a_i \geqslant 0$ for $i = n+1, \ldots, m$ and $a_i > 0$ for $i = 1, \ldots, n$. The proof follows the same steps as in the proofs of theorems 2 and 3.

4. Determination of existence conditions for Lyapunov functions

The determination of necessary and sufficient conditions for a general square matrix to be admissible is an unsolved problem. We present an algebraic procedure that, together with modern tools of algebraic computation, allows its implementation for a general matrix M involving only parameters of the system. The idea is to impose successively that a quadratic function in one of the variables is not positive:

$$\alpha y^2 + \beta y + \gamma \leqslant 0 \qquad \forall y \tag{25}$$

which is equivalent to

$$\alpha < 0 \qquad \text{and} \qquad \beta^2 - 4\alpha\gamma \leqslant 0$$

or

$$\alpha = \beta = 0 \qquad \text{and} \qquad \gamma \leqslant 0.$$

Since (3) is a quadratic form on the w_i', we group different terms according to their degree in one of the variables, say ω_1:

$$a_i M_{ij} w_i w_j = a_1 M_{11} w_1^2 + \beta(w_2, \ldots, w_m) w_1 + \gamma(w_2, \ldots, w_m) \leqslant 0. \tag{26}$$

Equation (26) is a quadratic algebraic inequality in w_1 and therefore, as explained above, one of the following sets of conditions must hold:

$$a_1 M_{11} < 0 \qquad \text{and} \qquad \beta(w_2, \ldots, w_m)^2 - 4a_1\gamma(w_2, \ldots, w_m) \leqslant 0 \tag{27}$$

or

$$\begin{aligned} &a_1 M_{11} = 0 \Rightarrow M_{11} = 0 \qquad \text{and} \qquad \beta(w_2, \ldots, w_m) = 0 \\ &\text{and} \qquad \gamma(w_2, \ldots, w_m) \leqslant 0. \end{aligned} \tag{28}$$

Conditions (27) and (28) are independent of w_1. The procedure can be iterated for these inequalities, now in the variables $w_2, \ldots, w_m$. At each step one set of conditions split in two, corresponding to the different possibilities to satisfy (25). Therefore, for an m-dimensional matrix M, corresponding to an m-dimensional LV system (1), we obtain 2^{m-1} independent sets of equation and inequality conditions, each set yielding different solutions for the parameters and the $a_i{}'$ such that M is admissible.

Note that this procedure can be handled in different ways either by changing the order in which the $w_i{}'$ are eliminated or by changing the order in which the quasi-monomials in (15) are defined. This results in an equivalent set of conditions, since the procedure is exhaustive. This is a new purely algebraic method to analyse the stability of general QP systems of differential equations.

5. New results for Lotka–Volterra systems

The existence of a Lyapunov function of form (4) depends strongly on the number of free parameters in the original system (1) or (15), and therefore on the number of free elements of M that can be adjusted such that the matrix is admissible. We also note that matrix M does not depend on the parameters λ_i or l_i in equations (1) and (15) and the existence conditions of a Lyapunov function are independent of their values. Now we show how a special reparametrization of the time variable 'mixes' these parameters with the remaining ones in M, resulting in different conditions on the parameters of the system, now involving its linear terms. Let us consider the differential time reparametrization:

$$\mathrm{d}t = \prod_{k=1}^{m} x_k^{-\tilde{B}_{\gamma k}}\,\mathrm{d}t' \qquad \text{for some} \quad k \in 1, \ldots, m \tag{29}$$

or $\mathrm{d}t = U_\gamma^{-1}\,\mathrm{d}t'$, restricted to those transformations that preserve the sign of t (this is ensured for instance if the variables x_i are restricted to the positive orthant or by using their absolute values). Also, if we restrict ourselves to the interior of an orthant, there will be no problem concerning singularities in (29). Let us consider first these transformations in QP systems. Inserting (29) in (9) we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t'} = x_i \tilde{A}_{ir} + x_i \left(\tilde{l}_i \prod_{k=1}^{m} x_k^{-\tilde{B}_{rk}} + \sum_{j \neq r}^{m} \tilde{A}_{ij} \prod_{k=1}^{m} x_k^{\tilde{B}_{jk} - \tilde{B}_{rk}} \right). \tag{30}$$

In this way we obtain a new QP system with different linear terms and with the same number of quasi-monomials (compare (30) with (9)). We can recast this system into the LV format, with a new characteristic matrix M', as described in section 3. By doing so we are able to obtain different existence conditions for a Lyapunov function.

This result can be used to obtain new results in purely LV systems. The classical method to analyse the stability of these systems consists in determining whether the corresponding matrix M is admissible, which implies the stability of its interior fixed points, with no dependence on the particular values of the parameters λ_i. We now show how in some cases it is possible to obtain Lyapunov functions for LV systems even when the characteristic matrix M is not admissible. This result is not unexpected from the theoretical point of view, since admissibility is a condition of sufficiency only. To prove this assertion we apply the transformation (29) in the LV system (1). Let $\mathrm{d}t = U_k^{-1}\,\mathrm{d}t'$, for some $k \in \{1, \ldots, m\}$; we obtain

$$\frac{\mathrm{d}U_i}{\mathrm{d}t'} = U_i \left(M_{ik} + \sum_{j \neq k}^{m} M_{ij} U_j U_k^{-1} + \lambda_i U_k^{-1} \right) \qquad i = 1, \ldots, m. \tag{31}$$

This reparametrization preserves the sign of the time variable provided the initial condition is in the interior of the positive orthant, since this orthant is invariant (Hofbauer 1988). System (31) is now an m-dimensional QP system which can be recast in an m-dimensional LV system by defining the following matrices B and A, from equation (31):

$$B = \begin{bmatrix} 0 & 0 & \ldots & -1 & \ldots & 0 \\ 1 & 0 & \ldots & -1 & \ldots & 0 \\ 0 & 1 & \ldots & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & -1 & \ldots & 1 \end{bmatrix} \tag{32}$$

with $[-1, -1, \ldots, -1]^T$ as the kth column;

$$A = \begin{bmatrix} \lambda_1 & M_{11} & M_{12} & \ldots & M_{1,k-1} & M_{1,k+1} & \ldots & M_{1m} \\ \lambda_2 & M_{21} & M_{22} & \ldots & M_{2,k-1} & M_{2,k+1} & \ldots & M_{2m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_m & M_{m1} & M_{m2} & \ldots & M_{m,k-1} & M_{m,k+1} & \ldots & M_{mm} \end{bmatrix}. \tag{33}$$

System (31) is equivalent to a LV system with characteristic matrix $\widetilde{M} = BA$ corresponding to the evolution of the quasi-monomials:

$$V_i = U_i/U_k \qquad \text{for} \quad i \neq k \quad V_k = 1/U_k. \tag{34}$$

The matrix $\widetilde{M}$ will involve the λ_i' and may be admissible even when M is not. This enlarges the possibilities of finding stable cases in LV systems, as a multitude of new time transformations can be defined. We presented above just an illustrative example of such transformations. We illustrate this result by analysing the May–Leonard system.

5.1. The May–Leonard system

The May–Leonard system describes three competing populations and was first studied in May and Leonard (1975):

$$\begin{aligned} \dot{x}_1 &= l_1 x_1 - x_1(x_1 + a x_2 + b x_3) \\ \dot{x}_2 &= l_2 x_2 - x_2(b x_1 + x_2 + a x_3) \\ \dot{x}_3 &= l_3 x_3 - x_3(a x_1 + b x_2 + x_3). \end{aligned} \tag{35}$$

In this case the variables x_i are identical to the variables U_i defined in section 2. Matrix M is given here by

$$M = \begin{bmatrix} -1 & -a & -b \\ -b & -1 & -a \\ -a & -b & -1 \end{bmatrix}. \tag{36}$$

The conditions for admissibility of this matrix are well known and are given by (see, e.g., Hofbauer and Sigmund (1988)):

$$-1 < a + b < 2 \qquad \forall\, l_i \in \mathbb{R}. \tag{37}$$

For the sake of brevity, we do not show here how this result can be obtained by using our procedure described in section 4. Instead we proceed to show how to obtain new conditions of stability outside the region given by (37).

Performing the time reparametrization $\mathrm{d}t = x_1^{-1}\,\mathrm{d}t'$, system (35) is written as

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t'} &= x_1(l_1 x_1^{-1} - 1 - a x_1^{-1} x_2 - b x_1^{-1} x_3) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t'} &= x_2(l_2 x_1^{-1} - b - x_1^{-1} x_2 - a x_1^{-1} x_3) \\ \frac{\mathrm{d}x_3}{\mathrm{d}t'} &= x_3(l_3 x_1^{-1} - a - b x_1^{-1} x_2 - x_1^{-1} x_3) \end{aligned} \tag{38}$$

which is a QP system with matrices B and A as follows:

$$B = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \tag{39}$$

$$A = \begin{bmatrix} l_1 & -a & -b \\ l_2 & -1 & -a \\ l_3 & -b & -1 \end{bmatrix}. \tag{40}$$

The characteristic matrix $\widetilde{M} = BA$ is given by

$$\widetilde{M} = \begin{bmatrix} -l_1 & a & b \\ l_2 - l_1 & a-1 & b-a \\ l_3 - l_1 & a-b & b-1 \end{bmatrix} \tag{41}$$

corresponding to the evolution of the quasi-monomials $V_1 = x_1^{-1}$, $V_2 = x_2 x_1^{-1}$ and $V_3 = x_3 x_1^{-1}$. Systems (35) and (38) are mathematically equivalent. Therefore, conditions for the admissibility of $\widetilde{M}$ imply the stability of the interior fixed points of (35), which are given by

$$\begin{aligned} q_1 &= \frac{(l_1(1-ba) + l_2(b^2-a) + l_3(a^2-b))}{(b^3+a^3-3ba+1)} \\ q_2 &= \frac{(l_2(1-ba) + l_3(b^2-a) + l_1(a^2-b))}{(b^3+a^3-3ba+1)} \\ q_3 &= \frac{(l_3(1-ba) + l_1(b^2-a) + l_2(a^2-b))}{(b^3+a^3-3ba+1)}. \end{aligned} \tag{42}$$

This leads to new cases of stability involving l_1 and l_2 and outside the region $-1 < a+b < 2$. Applying our procedure introduced in section 4, we obtain $2^{3-1} = 4$ independent sets of conditions for admissibility. Let us consider the most simple of these sets which gives, after some algebraic manipulations:

- Conditions 1

$$\begin{aligned} &l_1 = 0 \\ &a < 1 \\ &a_2 = a_1(-a/l_2) > 0 \qquad l_2 \neq 0 \\ &a_3 = a_1(-b/l_3) > 0 \qquad l_3 \neq 0 \\ &((a^2-ab)l_3 - (b^2-ab)l_2)^2 + 4abl_2l_3(a+b+ab-1-a^2-b^2) \leqslant 0. \end{aligned} \tag{43}$$

  A detailed analysis of the above conditions leads to the conclusion that they are always satisfied if, for example,

$$\begin{aligned} &l_1 = 0 \\ &a = b < 1 \\ &a/l_2 < 0 \qquad l_2 \neq 0 \\ &b/l_3 < 0 \qquad l_3 \neq 0. \end{aligned} \tag{44}$$

  By direct substitution, one can show that the above restrictions are sufficient for the last inequality in (44). Other sufficient conditions are given by

$$\begin{aligned} &l_1 = 0 \\ &a < 1 \\ &a/l_2 = b/l_3 < 0 \qquad l_2 \neq 0 \quad l_3 \neq 0. \end{aligned} \tag{45}$$

  Both the above sets of conditions allow values for a and b outside the range $-1 < a+b < 2$. Another set of conditions which is algebraically simple is the following:

- Conditions 2

$$\begin{aligned} &l_1 = 0 \qquad l_2 = l_3 \\ &a_2 = a_1(-a/l_2) > 0 \\ &a_3 = a_1(-b/l_3) > 0 \\ &b \leqslant 1 \qquad a = 1. \end{aligned} \tag{46}$$

  This set of conditions clearly allows values for a and b outside $-1 < a+b < 2$.

  The two further sets of conditions are a bit more complex algebraically and are given by

- Conditions 3

$$
\begin{aligned}
& a_1 = 1 \\
& l_1 > 0 \\
& 0 < \frac{2l_1 - a(l_1 + l_2) - 2\sqrt{\Omega_1}}{(l_2 - l_1)^2} < a_2 < \frac{2l_1 - a(l_1 + l_2) + 2\sqrt{\Omega_1}}{(l_2 - l_1)^2} \\
& \Omega_1 = l_1(a-1)(al_2 - l_1) > 0 \\
& [(l_1 - l_3)^2 a_3^2 + 2(bl_1 + bl_3 - 2l_1)a_3 + b^2][((1-b)l_1 + (b-a)l_2 + (a-1)l_3)^2 a_3^2 \\
& \qquad + (2l_3(1-a)(b-a^2) + 2l_2(2a - b^2 - 2a^2 - a^2 b + a^3 + 2b^2 a - ba) \\
& \qquad + 2l_1(2ba + b - 2 - b^2 - a^2 b + 2a - a^2))a_3 \\
& \qquad + (b - a^2)^2] \leqslant 0.
\end{aligned} \tag{47}
$$

- Conditions 4

$$
\begin{aligned}
& l_1 > 0 \\
& a_2 = a_1 \frac{2l_1 - a(l_1 + l_2) \pm 2\sqrt{l_1(a-1)(al_2 - l_1)}}{(l_2 - l_1)^2} > 0 \\
& a_3 = a_1\big(l_1^2(4a - 2a^2 - 2b) + l_1 l_2(4ab - 2a^2 - 2b) \pm 2(l_1(2a - b) - l_2 b)\sqrt{\Omega_1}\big) \\
& \qquad \times\big\{(l_1 a - 2l_1 b + al_3)(l_2 - l_1)^2 + 2l_1 l_2 l_3 + l_1^3(2-a) + l_2^2 a(l_1 - l_3) \\
& \qquad + l_1^2(l_3 a - 2l_3 - 2l_2) \pm 2\Omega_4\sqrt{\Omega_1}\big\}^{-1} > 0 \\
& \Omega_3^2(l_1 - l_3)^2 + 2\Omega_2\Omega_3(2l_1 - b(l_1 + l_3) + b^2\Omega_2^2 \leqslant 0
\end{aligned} \tag{48}
$$

where

$$
\begin{aligned}
\Omega_1 &= l_1(a-1)(al_2 - l_1) > 0 \\
\Omega_2 &= (l_1 a - 2l_1 b + al_3)(l_2 - l_1)^2 + 2l_1 l_2 l_3 + l_1^3(2-a) - l_2^2 l_3 a - 2l_1^2 l_2 \\
&\qquad + l_2^2 l_1 a - 2l_1^2 l_3 + l_1^2 l_3 a \pm 2(l_2 l_3 + l_1^2 - l_2 l_1 - l_1 l_3)\sqrt{\Omega_1} \\
\Omega_3 &= ab(l_2 - l_1)^2 + l_1^2(2a^2 - 4a + 2b - ba) - 2l_2^2 ba + 2l_1 l_2(a^2 - ab + b) \\
&\qquad \pm(-4al_1 + 2l_1 b + 2l_2 b)\sqrt{\Omega_1} \\
\Omega_4 &= -l_1 l_2 - l_1 l_3 + l_2 l_3 + l_1^2.
\end{aligned}
$$

The plus–minus sign in conditions 4 means that we have in fact two independent sets of conditions.

A detailed analysis of each one of the above conditions would not be suitable here. One can attempt to search for values of a and b outside the interval $-1 < a+b < 2$ with reasonable constraints in the l_i', or alternatively attempt to use other time reparametrizations than the one considered here.

The Lyapunov function is obtained from (4) with the monomials given by V_j in (34), which are related to (38), and therefore the corresponding fixed points are given by

$$
q_1^* = 1/q_1 \qquad q_2^* = q_2/q_1 \qquad \text{and} \qquad q_3^* = q_3/q_1
$$

with q_1, q_2 and q_3 given in (42). Explicitly, we have

$$
\begin{aligned}
\overline{V} = a_1\bigg(&\frac{1 + a_2 x_2 + a_3 x_3}{x_1} - q_1^* - a_2 q_2^* - a_3 q_3^* - q_1^* \ln(1/x_1 q_1^*) \\
&- a_2 q_2^* \ln(x_2/x_1 q_2^*) - a_3 q_3^* \ln(x_3/x_1 q_3^*)\bigg).
\end{aligned}
$$

Using the permutation symmetry of (35) one can obtain other stability conditions and Lyapunov functions by a suitable permutation of the indices and of the parameters a and b.

6. Application to a quasi-polynomial system

In this section we apply our approach to a system describing the nonlinear coupling of three modes in plasma waves.

6.1. Three-wave interaction problem

We consider here the problem of nonlinear interaction of three waves, which is an approximation for a general description of coupling in various fields of physics (Weiland and Wilhelmson 1977). The set of equations are given by

$$\begin{aligned}\dot{x}_1 &= \lambda_1 x_1 + x_1\left[\sum_{j=1}^{3} N_{1j}x_j^2\right] + \gamma_1 x_2 x_3\\ \dot{x}_2 &= \lambda_2 x_2 + x_2\left[\sum_{j=1}^{3} N_{2j}x_j^2\right] + \gamma_2 x_1 x_3\\ \dot{x}_3 &= \lambda_3 x_3 + x_3\left[\sum_{j=1}^{3} N_{3j}x_j^2\right] + \gamma_3 x_1 x_2\end{aligned} \tag{49}$$

where the $\gamma_i{}'$, $N_{ij}{}'$ and $\lambda_i{}'$ are real parameters. System (49) can be cast into a six-dimensional LV system. Let us consider the particular case $\gamma_2 = \gamma_3 = 0$, $\gamma_3 \equiv \gamma$. In this case, (49) can be recast into a four-dimensional LV system with quasi-monomial variables $U_1 = x_1^2$, $U_2 = x_2^2$, $U_3 = x_3^2$ and $U_4 = x_1^{-1}x_2x_3$. The matrix M is given by

$$M = \begin{bmatrix} 2N_{11} & 2N_{12} & 2N_{13} & 2\gamma \\ 2N_{21} & 2N_{22} & 2N_{23} & 0 \\ 2N_{31} & 2N_{32} & 2N_{33} & 0 \\ -N_{11}+N_{21}+N_{31} & -N_{12}+N_{22}+N_{32} & -N_{13}+N_{23}+N_{33} & -\gamma \end{bmatrix}. \tag{50}$$

Applying our method, one obtains $2^{4-1} = 8$ sets of conditions for the admissibility of (50). We present in table 1 some of these sets of conditions, leaving aside conditions algebraically too complex. When one of these sets of conditions hold, we obtain for system (49) a Lyapunov function, which in this case is given by

$$\begin{aligned}V = a_1\Bigg(&x_1^2 - q_1 \ln\frac{x_1^2}{q_1} - q_1 + f\left(x_2^2 - q_2\ln\frac{x_2^2}{q_2} - q_2\right) + g\left(x_3^2 - q_3\ln\frac{x_3^2}{q_3} - q_3\right)\\ &+h\left(x_2x_3/x_1 - q_4\ln\frac{x_2x_3}{x_1q_4} - q_4\right)\Bigg)\end{aligned}$$

where f, g and h are given in table 1 and the fixed points are given by

$$\begin{aligned} q_2 &= \frac{-q_1(N_{23}N_{31} - N_{33}N_{21}) + N_{33}\lambda_2 - \lambda_3 N_{23}}{N_{23}N_{32} - N_{33}N_{22}}\\ q_3 &= \frac{q_1(N_{22}N_{31} - N_{32}N_{21}) + N_{22}\lambda_3 - \lambda_2 N_{32}}{N_{23}N_{32} - N_{33}N_{22}}\\ q_4 &= -q_1(-N_{12}N_{23}N_{31} + N_{12}N_{33}N_{21} + N_{22}N_{13}N_{31}\\ &\qquad +N_{11}N_{23}N_{32} - N_{11}N_{33}N_{22} - N_{13}N_{21}N_{32} + (N_{12}N_{33} - N_{13}N_{32})\lambda_2\\ &\qquad -(N_{12}N_{23} - N_{22}N_{13})\lambda_3\\ &\qquad +(N_{23}N_{32} - N_{33}N_{22})\lambda_1)/\gamma_1(N_{23}N_{32} - N_{33}N_{22})) \end{aligned} \tag{51}$$

with q_1 arbitrary.

Table 1. Conditions of stability for the three-wave system with $\gamma_2 = \gamma_3 = 0$.

Conditions 1	$N_{11} = 0$	$a_2 = f a_1$
	$N_{22} < 0$	$a_3 = g a_1$
	$f = -N_{12}/N_{21} > 0$	$a_4 = h a_1$
	$g = -N_{13}/N_{31} > 0$	
	$h = -2\gamma/(N_{21} + N_{31}) > 0$	
	$N_{13}^2 N_{32}^2 N_{21}^2 - 4N_{12}N_{22}N_{13}N_{33}N_{31}N_{21} + N_{12}^2 N_{23}^2 N_{31}^2$	
	$+2N_{12}N_{23}N_{13}N_{32}N_{31}N_{21} = 0$	
	$-N_{12}N_{23}N_{32}N_{31} + N_{12}^2 N_{23}N_{31} + 2N_{12}N_{22}N_{33}N_{31} + N_{13}N_{32}N_{12}N_{21}$	
	$-N_{13}N_{32}N_{22}N_{21} + N_{12}N_{23}N_{22}N_{31} - N_{13}N_{32}^2 N_{21} - 2N_{12}N_{22}N_{13}N_{31} = 0$	
	$2N_{32}N_{22}N_{21} + 2N_{12}N_{22}N_{21} + N_{32}^2 N_{21} + N_{12}^2 N_{21} + N_{22}^2 N_{21}$	
	$+4N_{12}N_{22}N_{31} - 2N_{32}N_{12}N_{21} \leqslant 0$	
Conditions 2	$N_{11} = 0$	$a_2 = f a_1$
	$N_{22} < 0$	$a_3 = g a_1$
	$f = -N_{12}/N_{21} > 0$	$a_4 = h a_1$
	$g = -N_{13}/N_{31} > 0$	
	$h = -2\gamma/(N_{21} + N_{31}) > 0$	
	$N_{13}^2 N_{32}^2 N_{21}^2 - 4N_{12}N_{22}N_{13}N_{33}N_{31}N_{21}$	
	$+N_{12}^2 N_{23}^2 N_{31}^2 + 2N_{12}N_{23}N_{13}N_{32}N_{31}N_{21} < 0$	
	$N_{22}(-N_{13}^2 N_{32}^2 N_{21}^3 - N_{12}^2 N_{23}^2 N_{31}^3 - N_{12}N_{23}N_{32}N_{33}N_{21}N_{31}^2$	
	$-N_{23}N_{32}^2 N_{13}N_{21}^2 N_{31} - N_{12}N_{23}^2 N_{32}N_{21}N_{31}^2 - N_{12}N_{23}N_{32}N_{13}N_{21}N_{31}^2$	
	$+N_{12}^2 N_{23}N_{33}N_{21}N_{31}^2 - N_{12}N_{23}N_{32}N_{13}N_{21}^2 N_{31} - N_{12}^2 N_{23}N_{13}N_{21}N_{31}^2$	
	$-N_{12}N_{33}N_{13}N_{32}N_{21}^2 N_{31} + N_{22}N_{33}N_{13}N_{32}N_{21}^2 N_{31} + N_{12}N_{22}N_{33}N_{23}N_{21}N_{31}^2$	
	$+2N_{12}N_{22}N_{33}N_{13}N_{21}N_{31}^2 - N_{13}^2 N_{32}N_{12}N_{21}^2 N_{31} - N_{13}N_{32}N_{22}N_{23}N_{21}^2 N_{31}$	
	$+N_{13}^2 N_{32}N_{22}N_{21}^2 N_{31} - N_{12}N_{23}N_{22}N_{13}N_{21}N_{31}^2 + N_{12}N_{22}N_{13}^2 N_{21}N_{31}^2$	
	$+2N_{12}N_{22}N_{33}N_{13}N_{21}^2 N_{31} + N_{12}^2 N_{13}N_{33}N_{21}^2 N_{31} + N_{22}^2 N_{13}N_{33}N_{21}^2 N_{31}) \geqslant 0$	
Conditions 3	$N_{11} < 0$	$a_2 = f a_1$
	$f = 2N_{11}N_{22} - N_{12}N_{21} \pm \sqrt{N_{22}N_{11}(N_{11}N_{22} - N_{12}N_{21})}/N_{21}^2 > 0$	$a_3 = g a_1$
		$a_4 = h a_1$
	$g = (f[2N_{11}N_{23} - N_{21}N_{13}] - N_{13}N_{12})/(f N_{21}N_{31} - 2N_{11}N_{32} + N_{12}N_{31}) > 0$	
	$h = -2\gamma(gN_{31} + N_{13})/(N_{21}N_{13}$	
	$+N_{13}N_{11} - 2N_{11}N_{23} + N_{31}N_{13} - 2N_{11}N_{33}$	
	$+g[N_{21}N_{31} + N_{31}^2 - N_{31}N_{11}]) > 0$	
	$2\gamma(N_{12} + f N_{21}) + h(-2N_{11}N_{22} + N_{12}[N_{11} + N_{21} + N_{31}] - 2N_{11}N_{32})$	
	$+f h(N_{21}^2 N_{21}N_{31} - N_{21}N_{11}) = 0$	
	$N_{13}^2 + (2N_{31}N_{13} - 4N_{11}N_{33})g + g^2 N_{31}^2 = 0$	
	$4\gamma^2 + (N_{21} + N_{11} + N_{31})4\gamma h + ((N_{31} - N_{11})^2 + N_{21}[N_{21} + N_{31} - 2N_{11}])h^2 \leqslant 0$	

Conditions involving the $\lambda_i{}'$ can be obtained by performing a reparametrization in the time variable. In table 2 we present some results obtained from the specific case $\mathrm{d}t = U_1^{-1}\,\mathrm{d}t'$ and the corresponding Lyapunov function.

Table 2. Conditions for the three-wave system with the corresponding Lyapunov functions with $\mathrm{d}t = U_1^{-1}\,\mathrm{d}t'$.

Conditions 1	$\lambda_1 = 0$ $f = N_{12}/\lambda_2 > 0$ $g = N_{13}/\lambda_3 > 0$ $h = 2\gamma/\lambda_2 + \lambda_3 > 0$ $N_{22} = N_{12}$ $N_{33} = N_{13}$ $\gamma = \dfrac{(N_{23} - 3N_{13})(-N_{32} + 2N_{12})}{N_{12}(N_{23} - 2N_{13})} \geqslant 0$ $\lambda_2 = \dfrac{(N_{23} - 3N_{13})\lambda_3}{N_{13}}$ $2N_{12}N_{13} + N_{32}N_{23} - 3N_{13}N_{32} = 0$	$a_2 = fa_1$ $a_3 = ga_1$ $a_4 = ha_1$
Conditions 2	$\lambda_1 = 0$ $f = N_{12}/\lambda_2 > 0$ $g = N_{13}/\lambda_3 > 0$ $h = 2\gamma/\lambda_2 + \lambda_3 > 0$ $N_{22} - N_{12} = 0$ $\lambda_3 N_{12}(N_{23} - N_{13}) - \lambda_2(N_{12} - N_{32})N_{13} = 0$ $-\gamma N_{12}(\lambda_2 + \lambda_3) - \lambda_2(N_{32} + N_{22} - 3N_{12}) = 0$ $N_{33} - N_{13} < 0$ $(4N_{13}^2 + 4N_{13}N_{33} - 8N_{13}N_{23} + N_{33}^2 + 2N_{33}N_{23} + N_{23}^2)\lambda_3^2$ $+(-4N_{13}^2 + 10N_{13}N_{33} - 2N_{13}N_{23})\lambda_2\lambda_3 + N_{13}^2\lambda_2^2 \leqslant 0$	$a_2 = fa_1$ $a_3 = ga_1$ $a_4 = ha_1$
Conditions 3	$\lambda_1 > 0$ $f = \dfrac{\lambda_1(N_{12} - 2N_{22}) + N_{12}\lambda_2 \pm 2\sqrt{\lambda_1(N_{12} - N_{22})(N_{12}\lambda_2 - \lambda_1 N_{22})}}{(\lambda_1 - \lambda_2)^2} > 0$ $g = \dfrac{f(\lambda_1(2N_{23} - N_{13}) - \lambda_2 N_{13} + N_{12}N_{13})}{f(\lambda_1(\lambda_2 + \lambda_3 - \lambda_1) - \lambda_2\lambda_3) + N_{12}\lambda_3 + N_{12}\lambda_1 - 2\lambda_1 N_{32})} > 0$ $h = 2\gamma(f(-\lambda_1 - \lambda_2) + N_{12})/(f(\lambda_1\lambda_3 + 4\lambda_1\lambda_2 - 3\lambda_1^2 - \lambda_2\lambda_3 - \lambda_2^2)$ $+N_{12}(3\lambda_1 + \lambda_3 + \lambda_2) - 2\lambda_1 N_{32} - 2\lambda_1 N_{22} > 0$ $g^2(\lambda_1 - \lambda_3)^2 + 2g(\lambda_1(2N_{33} - N_{13}) - \lambda_3 N_{13}) + N_{13}^2 = 0$ $g(h(\lambda_3^2 + 3\lambda_1^2 + \lambda_2\lambda_3 - 4\lambda_1\lambda_3 - \lambda_2\lambda_1) - 2\lambda_3\gamma - 2\lambda_1\gamma)$ $+(2\lambda_1[N_{23} + N_{33}) + N_{13}(-\lambda_2 - \lambda_3 - 3\lambda_1))h + 2N_{13}\gamma = 0$ $h^2((\lambda_2 + \lambda_3)^2 + 3\lambda_1(3\lambda_1 - 2\lambda_2 - 2\lambda_3)) + 4h\gamma(-\lambda_3 - 3\lambda_1 - \lambda_2) + 4\gamma^2 \leqslant 0$	$a_2 = fa_1$ $a_3 = ga_1$ $a_4 = ha_1$
Lyapunov function	$V = a_1((1 + fx_2^2 + gx_3^2 + hx_2x_3x_1^{-1})/x_1^2$ $-q_1^* - fq_2^* - gq_3^* - hq_4^* - q_1^*\ln(1/x_1^2q_1^*)$ $-fq_2^*\ln(x_2^2/x_1^2q_2^*)$ $-gq_3^*\ln(x_3^2/x_1^2q_3^*) - hq_4^*\ln(x_2x_3/x_1^3q_4^*))$ $q_1^* = 1/q_1;\ \ q_2^* = q_2/q_1;\ \ q_3^* = q_3/q_1;\ \ q_4^* = q_4/q_1$	

We note that the conditions in table 1 involve only the parameters N_{ij}. Therefore, the conditions of stability will prevail whatever the particular values of the linear parameters λ_i.

In table 2 the parameters N_{11}, N_{21} and N_{31} are not present, but the conditions involve the λ_i', as, for example, the first condition from table 2.

Several conditions for the general case $\gamma_i \neq 0$ were obtained, but are not shown here for reasons of economy of space.

7. Numerical method

The method described in section 4 is suitable for analysing matrices M whose parameters are given algebraically. However, in many cases the numerical values of the parameters of the system are known, or are at least limited to a small range. In such cases it is suitable to present a numerical version of our approach. Specifically, we address the following question: given a real matrix M, with all its entries specified, determine if such matrix is admissible or not and, if the answer is affirmative, provide the set of numbers $a_i > 0$. The authors have developed such an algorithm, which will be the subject of a forthcoming paper.

This approach is not a numerical version of the algebraic method described in section 4. Indeed, it makes use of techniques very usual in system and control theory, and is based on linear programming.

8. Conclusions

The universal format of the LV equations provides a powerful tool for obtaining new methods for the study of arbitrary nonlinear systems (Moreau *et al* 1999). The pioneering work of Volterra showed the importance of the concept of admissible matrices in the study of the stability problem, which was later extended by other authors. In this paper we have extended the analysis to general QP systems, which is an attempt towards a universal method for the determination of Lyapunov functions applicable to a broad class of ordinary differential equations.

Furthermore, the time reparametrization (29) increases the scope of applications even for purely LV systems, as shown by the new conditions of stability obtained for the May–Leonard system, involving its linear terms, improving the known results in the literature for this system. Our approach has no restrictions concerning the dimension of the system. However, when the conditions obtained are too complex, a numerical algorithm for solving (3) is needed. This procedure is well suited for larger systems and will be the subject of a forthcoming paper.

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References

Brenig L 1988 Complete factorization and analytic solutions of generalized Lotka–Volterra equations *Phys. Lett.* A **133** 378–82

Brenig L and Goriely A 1989 Universal canonical formats for time-continuous dynamical systems *Phys. Rev.* A **40** 4119–22

Figueiredo A, Gléria I M and Rocha Filho T M 2000 Boundedness of solutions and Lyapunov functions in quasi-polynomial dynamical systems *Phys. Lett.* A **268** 335–41

Figueiredo A, Rocha Filho T M and Brenig L 1998 Algebraic structures and invariants manifolds of differential systems *J. Math. Phys.* **39** 2929–46

——1999 Necessary conditions for the existence of quasi-polynomial invariants: the quasi-polynomial and Lotka–Volterra systems *Physica* A **262** 158–79

Haykin S 1999 *Neural Networks—A Comprehensive Foundation* (Englewood Cliffs, NJ: Prentice-Hall)

Hofbauer J and Sigmund K 1988 *The Theory of Evolution and Dynamical Systems* (Cambridge: Cambridge University Press)

LaSalle J P and Lefshets S 1961 *Stability by Lyapunov Direct Method* (New York: Academic)

Lyapunov A M 1949 Problème géneral de la stabilité du mouvement *Ann. Math. Stud.* vol 17 (Princeton: Princeton University Press)

May R and Leonard W J 1975 Nonlinear aspects of competition between three species *SIAM J. Appl. Math.* **29** 243–53

McLachan R I, Quispel G R W and Robidoux N 1998 Unified approach to Hamiltonian systems, gradient systems, and systems with Lyapunov functions or first integrals *Phys. Rev. Lett.* **81** 2399–403

Moreau Y, Loueis S, Vandewalle J and Brenig L 1999 Embedding recurrent neural-networks into prey-predator models *Neural Networks* **12** 237–45

Redheffer R and Walter W 1984 Solution of the stability problem for a class of generalized Volterra prey-predator systems *J. Diff. Eqns* **52** 245–63

Takeuchi Y 1995 *Global Analitical Properties of Lotka–Volterra Systems* (London: World Scientific)

Volterra V 1931 *Leçons Sur La Théorie Mathématique De La Lutte Pour La Vie* (Paris: Gauthier-Villars)

Weiland J and Wilhelmsson H 1977 *Coherent Nonlinear Interaction of Waves in Plasmas* (Oxford: Pergamon)